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Brief paper

Distributed finite-time consensus of nonlinear systems under switching topologies ☆

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ABSTRACT

In this paper, the finite time consensus problem of distributed nonlinear systems is studied under the general setting of directed and switching topologies. Specifically, a contraction mapping argument is used to investigate performance of networked control systems, two classes of varying topologies are considered, and distributive control designs are presented to guarantee finite time consensus. The proposed control scheme employs a distributed observer to estimate the first left eigenvector of graph Laplacian and, by exploiting this knowledge of network connectivity, it can handle switching topologies. The proposed methodology ensures finite time convergence to consensus under varying topologies of either having a globally reachable node or being jointly strongly connected, and the topological requirements are less restrictive than those in the existing results. Numerical examples are provided to illustrate the effectiveness of the proposed scheme.

1. Introduction

Distributed consensus is a study dedicated to ensuring an agreement between states or output variables among networked systems (Qu, 2009). It is well established that common challenges in this venue are how to achieve consensus with the least possible topological requirement, and how to achieve it in a timely and distributive manner. In this regard, distributed finite time consensus became an instant popular topic in the community, especially with recent advances on finite time stability (Bhat & Bernstein, 2000). Breakthroughs have been made with both continuous (Jiang & Wang, 2009; Khoo, Xie, & Man, 2009; Li, Du, & Lin, 2011; Ou, Du, & Li, 2014; Shang, 2012; Wang & Xiao, 2010; Xiao, Wang, Chen, & Gao, 2009) and discontinuous inputs (Cao & Ren, 2012a,b; Chen, Lewis, & Xie, 2011; Cortés, 2006; Shi & Hong, 2009; Sundaram & Hadjicostis, 2007). To be more precise, finite time consensus with continuous input can be treated as an extension of Bhat and Bernstein (2000) to multi-agent systems, and are in general conducted under time-invariant graph. In particular, it is shown in Jiang and Wang (2009); Wang and Xiao (2010); Xiao et al. (2009) that the graph shall be undirected or directed but detailed-balanced in order to achieve a finite time convergence. This condition is further released in Shang (2012), where finite time consensus is ensured for digraph (i.e., directed graph) with a spanning tree. In addition, applications of continuous finite time consensus have been carried out in formation control of leader–follower multi-agent systems (Li et al., 2011) and nonholonomic robots (Ou et al., 2014), as well as robust finite time tracking of multi-robot systems (Khoo et al., 2009).

Due to the highly nonlinear nature of discontinuous input (i.e, signum/binary protocol), the convergence analysis of networked systems with discontinuous input is extremely challenging, and its solution, if possible, is often sophisticated. For instance, Cortés (2006) pioneered finite time consensus with discontinuous input, under undirected graph, using nonsmooth stability analysis. In Chen et al. (2011), Filippov solution (Filippov, 1960) is introduced for undirected and directed but detailed-balanced networks using a binary protocol and pinning control scheme. The most recent contribution for this topic witnessed the application of a comparison-based Lyapunov approach in both directed (Cao & Ren, 2012b) and undirected (Cao & Ren, 2012a) networks. In addition,

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discrete finite time consensus of time-invariant network is investigated in Sundaram and Hadjicostis (2007) using weighted matrix with minimal polynomial of the smallest degree. Additionally, Shi and Hong (2009) focuses on directed and switching network, and it proves that digraph shall be quasi-strongly connected and contain no direct circle at any interval, in order to ensure a finite time convergence.

However, it should be pointed out that all of the aforementioned results are derived with rather restrictive topological requirements (i.e., undirected graph, digraph but detailed-balanced, or digraph being quasi-strongly connected), finite time consensus of a generic directed network with switching topologies has not received sufficient attention. In this paper, we attempt to solve this problem for a class of nonlinear systems under mild assumptions. The main contribution of this paper is twofold: (i) what are the least conservative topological requirements to ensure a finite time consensus under directed and switching topologies? Is there a simple argument to perform the convergence study of networked systems with discontinuous input? In this paper, we attempt to provide answers to these two questions; and (ii) with the recent advance on network connectivity of a digraph (Qu, Li, & Lewis, 2014), we propose a distributive control scheme that makes finite time consensus possible over any jointly strongly connected network.

2. Preliminaries on graph theory

In this paper, we consider a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E}$ denote the set of vertices/nodes and the set of directed edges/paths, respectively. Vertex j is said to be adjacent to vertex i if there exists a directed edge $(j, i) \in \mathcal{E}$ with node i being the head and node j being the tail. Analogously, neighborhood set $\mathcal{N}_i \subseteq \mathcal{V}$ of vertex i is $\{k \in \mathcal{V} \mid (k, i) \in \mathcal{E}\}$. Without loss of any generality, adjacency matrix $A(\mathcal{D})$ used in this paper is weighted and normalized as:

$$[A(\mathcal{D})]_{ik} = \begin{cases} a_{ik} > 0 & \text{if } k \neq i,\ (k, i) \in \mathcal{E} \\ 1 - \sum\limits_{k \neq i} a_{ik} & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

That is, matrix $A(\mathcal{D})$ is chosen to be nonnegative and row-stochastic. Furthermore, we assume that the nonzero, and hence positive, weighting factors are all uniformly lower and upper bounded as $\underline{a} \leq a_{ij} \leq 1$, where $0 < \underline{a} \leq 1$, for any $j \in \mathcal{N}_i$. As such, the weighted graph Laplacian is

$$\mathcal{L}(\mathcal{D}) \triangleq I - A(\mathcal{D}) \tag{2}$$

where I is the identity matrix with proper dimension.

Graph $\mathcal{D}$ is said to have *one globally reachable node* if there exists node i such that there is a directed path from node i to node j for all $j \in \mathcal{V}$ with $j \neq i$. Graph $\mathcal{D}$ is called *strongly connected* if there is a directed path between any pair of vertices: a directed path exists from i to k and so does a directed path from k to i for every pair of vertices i, k; or every node is a globally reachable node; or equivalently, Laplacian $\mathcal{L}(\mathcal{D})$ is *irreducible* (Qu, 2009).

According to (2), row sums of Laplacian $\mathcal{L}(\mathcal{D})$ are all zero. It follows that $\lambda_1 = 0$ is the smallest eigenvalue of $\mathcal{L}$ with right eigenvector $\nu_1 \triangleq \frac{1}{\sqrt{n}}\mathbf{1}_n$ and left eigenvector $\gamma = [\gamma_i] \in \Re^n$ defined by

$$\mathcal{L}^T\gamma = 0, \qquad \mathbf{1}_n^T\gamma = 1, \tag{3}$$

where $\mathbf{1}_n \triangleq [1 \ldots 1]^T$, and superscript T denotes matrix transpose. By Perron–Frobenius theorem, all other eigenvalues have positive real parts if the topology of graph $\mathcal{D}$ either has a globally reachable node or is strongly connected. Moreover, as shown in Qu et al. (2014), connectivity (and social standings in the connected network) of $\mathcal{D}$ can be described by left eigenvector γ (and its components). Existing results on γ and its distributed estimation are summarized into the following lemma; its proof is omitted here since it merely combines the results in Qu (2009); Qu et al. (2014).

Lemma 1. *Consider graph Laplacian $\mathcal{L}$ defined by (2) with γ being its left eigenvector defined in (3). Then, the following results hold:*

- *if $\mathcal{D}$ has a global reachable node, γ is unique and non-negative, and $\gamma_i > 0$ (e.g., $\gamma_i = 1$) implies that node i belongs to the leader group[2] (e.g., being a sole leader), and $\gamma_i = 0$ means that node i belongs to the follower group. If $\mathcal{D}$ is strongly connected, $\gamma_i > 0$ for all i;*
- *γ can be estimated distributively at system i by*

$$\dot{\hat{\gamma}}^{(i)}(t) = \sum_{j=1}^{n} a_{ij}(t)\left[\hat{\gamma}^{(j)}(t) - \hat{\gamma}^{(i)}(t)\right] \tag{4}$$

where $\hat{\gamma}^{(i)} \in \Re^n$ is the estimate of γ at system i, $\hat{\gamma}^{(i)}(t_0) = e_i$, $e_i \in \Re^n$ is a vector of zeros except its ith entry being one, and $a_{ij}(t)$ are those defined in (1). Note that $\hat{\gamma}^{(i)}(t) = e_i$ must be reset once any topological switching is detected locally (by examining its corresponding row components of $\mathcal{L}$) and that such resetting should be propagated to the neighbors.

The following lemma will be used in the subsequent technical derivations.

Lemma 2. *Consider Laplacian matrix $\mathcal{L}$ defined in (2) and its left eigenvector γ defined in (3). Then, for any $\mu > 0$ and $t > 0$ and for any $\mathcal{D}$ with a globally reachable node,*

$$e^{\mp\mu\mathcal{L}t} = \mathbf{1}_n\gamma^T + \Gamma_s e^{\mp\mu\Lambda_s t}W_s^T,$$

where Λ_s is the Jordan form associated with eigenvalues λ_2 up to λ_n, $\Gamma_s \in \Re^{n\times(n-1)}$ is the resulting matrix of corresponding right eigenvectors after removing eigenvector $\mathbf{1}_n$ associated with $\lambda_1(\mathcal{L}) = 0$, $W_s \in \Re^{n\times(n-1)}$ consists of all the left eigenvectors of A except for γ.

To consider time-varying topologies, we introduce time sequence $\{t_k : k \in \aleph^+\}$ for $\aleph^+ = \{0, 1, \ldots, \infty\}$, and, without loss of any generality, graph $\mathcal{D}(t)$ is time invariant during interval $t \in [t_k, t_{k+1})$, that is, $A(t_k^+) = A(t_{k+1}^-)$.

3. Finite time consensus under switching topologies with globally reachable node(s)

Consider the network control problem for n nonlinear systems of identical dynamics:

$$\dot{x}_i = f(t, x_i) + u_i, \quad i \in \mathcal{V}, \tag{5}$$

where $x_i \in \Re^m$ is the state of the ith system, $u_i \in \Re^m$ is the neighboring feedback control to be designed, and $f(t, x_i)$ denotes the individual dynamics. For simplicity, $m = 1$ is set in the subsequent technical discussion, and the general case of $m > 1$ can be addressed analogously.

Function $f(t, x_i)$ is assumed to be uniformly bounded with respect to t and locally uniformly bounded with respect to x_i. It is obvious that system (5) is stabilizable, and hence it can be assumed without loss of any generality that, for all $x_i(0) \in \Omega_0$ and with $u_i \equiv 0$, $x_i(t)$ is uniformly bounded as $x_i(t) \in \Omega$ and

$$\|f(t, x_i)\| \leq \xi_f, \tag{6}$$

[2] Node i is said to be a leader (or belong to the leader group) if all edges initiated at node i are tails (or for any $j \to i$, node j is also a leader), node i being a follower can be defined analogously.

where $\xi_f > 0$ is a constant, Ω_0 is the (compact) set of interest, and $\Omega = \{x_i : |x_i| \le \xi_x\}$ for some constant $\xi_x > 0$. It is worth noting that, for synchronization of nonlinear oscillators, $\Omega_0 = \Re$.[3] The following lemma establishes uniform boundedness under a signum cooperative control.

Lemma 3. *Consider the systems in (5) and under the signum protocol*

$$u_i = \alpha \mathrm{sgn}\left\{\sum_{j=1}^{n} a_{ij}(t)[x_j - x_i]\right\}, \tag{7}$$

where $a_{ij}(t)$ are the entries of adjacency matrix A defined in (1), $\alpha > 0$ is a control gain, and $\mathrm{sgn}(\cdot)$ denotes the classical sign function. Then, $x_i(0) \in \Omega_0$ implies both $x_i(t) \in \Omega$ and inequality (6).

Proof. Define the indices sets associated with the maximal and minimal state variables as:

$$\begin{aligned} \overline{\mathcal{V}}(t) &= \left\{i \in \mathcal{V} :\ x_i(t) = \max_{j\in\mathcal{V}} x_j(t)\right\} \\ \underline{\mathcal{V}}(t) &= \left\{i \in \mathcal{V} :\ x_i(t) = \min_{j\in\mathcal{V}} x_j(t)\right\}. \end{aligned} \tag{8}$$

It follows that, for any $i \in \overline{\mathcal{V}}(t)$, $u_i \le 0$ and in turn $\dot{x}_i \le f(t, x_i)$. Applying comparison theorem (Khalil, 2002), it follows the maximum is preserved.

Similarly, the minimum is also preserved. Therefore, $x_i(t) \in \Omega$ and inequality (6) hold. □

The following lemma provides a result on contraction mapping and finite time convergence, and its proof is trivial and thus omitted for the sake of brevity.

Lemma 4. *Consider the systems in (5). If u_i are chosen such that the maximum disagreement among the state variables, denoted by*

$$\delta(t) \triangleq x_{i^*}(t) - x_{j^*}(t), \quad i^* \in \overline{\mathcal{V}}(t), j^* \in \underline{\mathcal{V}}(t), \tag{9}$$

has the properties that $\dot{\delta}(t) < 0$ and $|\dot{\delta}(t)| > \varepsilon$ for all time and for some constant $\varepsilon > 0$, then finite-time consensus is ensured with convergent time upper bounded by $\delta(t_0)/\varepsilon$.

The following theorem provides finite-time consensus by using the argument of contraction mapping.

Theorem 1. *Consider the systems in (5) and under cooperative control (7). If time-varying digraphs $\mathcal{D}(t)$ have at least one globally reachable node at every instant of time, then finite-time consensus can be achieved by choosing $\alpha > 2\xi_f$, and the convergence time is upper bounded by $(n-1)\delta(t_0)/(\alpha - 2\xi)$.*

Proof. It follows from (7), (8) and Lemma 3 that, for any $i^* \in \overline{\mathcal{V}}(t)$,

$$\dot{x}_{i^*} = \begin{cases} f(t, x_{i^*}) & \text{if } \gamma_{i^*} = 1, \text{ i.e., sole leader} \\ f(t, x_{i^*}) & \text{if } j \in \mathcal{N}_{i^*} \Longrightarrow j \in \overline{\mathcal{V}}(t) \\ f(t, x_{i^*}) - \alpha & \text{if otherwise.} \end{cases} \tag{10}$$

A similar expression can be obtained for $\dot{x}_{j^*}$ with $j^* \in \underline{\mathcal{V}}(t)$. Hence, we have for any pair of $i^* \in \overline{\mathcal{V}}(t)$ and $j^* \in \underline{\mathcal{V}}(t)$,

$$\dot{\delta} = \begin{cases} f(t, x_{i^*}) - f(t, x_{j^*}) & \text{if condition NC holds} \\ \begin{cases} f(t, x_{i^*}) - f(t, x_{j^*}) - \alpha \\ f(t, x_{i^*}) - f(t, x_{j^*}) - 2\alpha \end{cases} & \text{if otherwise,} \end{cases} \tag{11}$$

where condition NC (non-convergence) is defined by

$$\begin{cases} \text{either } \gamma_{i^*} = 1 \ \&\ (l \in \mathcal{N}_{j^*} \Longrightarrow l \in \underline{\mathcal{V}}(t)) \\ \text{or } \gamma_{j^*} = 1 \ \&\ (l \in \mathcal{N}_{i^*} \Longrightarrow l \in \overline{\mathcal{V}}(t)) \\ \text{or } (l \in \mathcal{N}_{i^*} \Longrightarrow l \in \overline{\mathcal{V}}(t)) \ \&\ (l \in \mathcal{N}_{j^*} \Longrightarrow l \in \underline{\mathcal{V}}(t)). \end{cases}$$

The requirement of $(l \in \mathcal{N}_{i^*} \Longrightarrow l \in \overline{\mathcal{V}}(t))$ says that only those state variables of the maximum value are connected to system i^* and, given that $\mathcal{D}$ has at least one globally reachable node (already specified by j^*), the second line in condition NC cannot hold for all $i^* \in \overline{\mathcal{V}}(t)$ unless $\delta(t) = 0$. This means that set $\overline{\mathcal{V}}(t)$ will decrease (entry by entry at the worst case), and the second line in condition NC becomes invalid for all i^* after an arbitrarily small period of time. Similarly, one can argue that the other two lines of condition NC will also become invalid. Therefore, we have

$$\dot{\delta}(t) \le |f(t, x_{i^*})| + |f(t, x_{j^*})| - \alpha \le 2\xi_f - \alpha < 0, \tag{12}$$

for any choice of $\alpha > 2\xi_f$. Finite-time consensus can then be concluded by invoking Lemmas 3 and 4. □

The contraction mapping argument used in the proof of Theorem 1 is straightforward in handling varying topologies and establishing finite-time convergence rate. The resulting topological condition of digraphs having at least one globally reachable node is less restrictive than those in the literature. For instance, the requirement in Cortés (2006); Wang and Xiao (2010) is that the graphs are undirected, and it is assumed in Chen et al. (2011); Shi and Hong (2009) that directed graphs are either detailed-balanced or quasi-strongly connected.

Note that Lemma 4 by itself is conservative because $\dot{\delta}(t) < 0$ is demanded for all t. As a result, Theorem 1 requires that all the varying topologies are individually connected in the sense that every topology has at least one globally reachable node, and its proof involves the use of relevant properties of γ_i. Nonetheless, Lemma 4 is applied again in the next section to establish finite-time consensus under the further relaxation that varying topologies do not necessarily have a globally reachable node but are jointly strongly connected.

4. Finite-time consensus under jointly strongly connected topologies

The following assumption is introduced to describe jointly strongly connected topologies.

Assumption 1a. Given time sequence $\{t_k : k \in \aleph^+\}$, varying digraphs $\mathcal{D}(t_k)$ are *jointly strongly connected*; that is, there is some known constant $\overline{T} > 0$ such that, over time intervals $[l\overline{T},\ (l+1)\overline{T})$ for $l \in \aleph^+$, the composite graph $\mathcal{D}_l$ (whose edges are the union of $\mathcal{E}(t_k)$ for all $t_k \in [l\overline{T},\ (l+1)\overline{T})$) is strongly connected.

Consider now the following distributed observer that records the maximum and minimum values of the local state variables: at system i,

$$\begin{cases} \overline{x}_i(t) = \max\left[\overline{x}_i(t^-),\ \max_{j\in\mathcal{N}_i} x_j(t)\right], \\ \underline{x}_i(t) = \min\left[\underline{x}_i(t^-),\ \min_{j\in\mathcal{N}_i} x_j(t)\right], \end{cases} \tag{13}$$

where $\overline{x}_i(0) = \underline{x}_i(0) = x_i(0)$. It follows that, as long as $x_j(t)$ stays bounded by $\max_j x_j(0)$ from above and by $\min_j x_j(0)$ from below, $\overline{x}_i(t)$ is monotonely increasing and upper bounded by $\max_j x_j(0)$ and $\underline{x}_i(t)$ is monotonely decreasing and lower bounded by $\min_j x_j(0)$. For any strongly connected graph, $\overline{x}_i(t)$ and $\underline{x}_i(t)$ reach the maximum and minimum values in the graph, respectively and instantaneously. Hence, we know from Assumption 1a that both

[3] We owe this observation to an anonymous reviewer.

$\overline{x}_i(t)$ and $\underline{x}_i(t)$ converge in a finite time (no later than $(n-1)\overline{T}$) to their consensus values, respectively.

The following distributed control assumes the knowledge of left eigenvector γ: at system i,

$$u_i = \begin{cases} \alpha \mathrm{sgn}\left(\dfrac{\overline{x}_i + \underline{x}_i}{2} - x_i\right) & \text{if condition IC holds} \\ \alpha \mathrm{sgn}\left(\displaystyle\sum_{j=1}^{n} a_{ij}(t)\left[x_j - x_i\right]\right) & \text{if otherwise} \end{cases} \tag{14}$$

where $a_{ij}(t)$ are the entries defined in (1), and condition IC (isolation condition) is defined by: at the ith system,

$$\text{condition IC} = \begin{cases} \text{either } \gamma_i = 1 \quad (i.e., \mathcal{N}_i = \emptyset) \\ \text{or } x_\ell = x_i, \quad \forall \ell \in \mathcal{N}_i. \end{cases} \tag{15}$$

Condition IC is introduced to ensure that, under control (14), an isolated/leading system or a system whose neighbors all have the equal value would have its state be moved toward the average of $(\overline{x}_i + \underline{x}_i)/2$ (which is a fixed point after a finite time). It is due to this choice of condition IC (which requires the knowledge of γ_i) that non-convergence phenomena such as oscillations and stalling are avoided under jointly strongly connected topologies (which individually may not have a globally reachable node). Essentially, alternating equilibria or isolated equilibria (that could exist under control (7) when varying topologies individually do not have a global reachable node) cannot exist under control (14). The finite-time consensus result is provided by the following theorem.

Theorem 2. *Consider the systems in (5) and under cooperative control (14) with $\alpha > 2\xi_f$. Suppose that varying topologies satisfy Assumption 1a and that the ith system has the local knowledge of γ_i, the ith component of left eigenvector γ defined in (3). Then, $x_i(t)$ converges to $(\overline{x}_i + \underline{x}_i)/2$ within a finite time no large than $(n-1)\overline{T} + (n-1)\delta(t_0)/(\alpha - 2\xi_f)$.*

Proof. Given (13), we know that $(\overline{x}_i + \underline{x}_i)/2$ reaches its consensus after finite time instant $t_1 \le (n-1)\overline{T}$, that is, there is a constant c_a such that

$$[\overline{x}_i(t) + \underline{x}_i(t)]/2 = c_a \quad \forall i \in \mathcal{V} \text{ and } \forall t \ge t_1.$$

Using the same argument in the proof of Lemma 3, we know that, under control (14), $x_i(t) \in [\min_i x_i(0),\ \max_i x_i(0)]$. It follows from (14) and (8) that, for any $i^* \in \overline{\mathcal{V}}(t)$ and for any $t \ge t_1$,

$$\dot{x}_{i^*} = \begin{cases} f(t, x_{i^*}) & \text{if (15) and } x_{i^*} = c_a \text{ hold} \\ f(t, x_{i^*}) + \alpha & \text{if (15) and } x_{i^*} < c_a \text{ hold} \\ f(t, x_{i^*}) - \alpha & \text{if (15) and } x_{i^*} > c_a \text{ hold} \\ f(t, x_{i^*}) - \alpha & \text{if (15) does not hold.} \end{cases}$$

A similar expression can be obtained for $\dot{x}_{j^*}$ with $j^* \in \underline{\mathcal{V}}(t)$. Consequently, we know from the above expression and from graphs being jointly strongly connected that, unless x_{i^*} (or x_{j^*}) has already reached c_a, i^* (or j^*) satisfying (15) means finite time convergence of $\dot{x}_{i^*}$ (or $\dot{x}_{j^*}$) to c_a. On the other hand, if both i^* and j^* do not satisfy (15), we have

$$\dot{\delta} = f(t, x_{i^*}) - f(t, x_{j^*}) - 2\alpha,$$

in which case finite time convergence to a consensus is ensured under the choice of $\alpha > 2\xi_f$. Once such a consensus is reached, condition (15) is met and finite-time convergence to c_a is under way. The proof is completed by invoking Lemma 4. □

It is worth noting that the choice of $(\overline{x}_i + \underline{x}_i)/2$ in (14) can be replaced by any fixed convex combination of $\overline{x}_i$ and $\underline{x}_i$ for all i and that, while the existence of $\overline{T}$ is required, its knowledge is not required unless we need to predict the amount of time needed to achieve the finite-time convergence. Performance of the proposed finite-time convergence protocol is illustrated by the following simple example.

Example 1. Consider the consensus problem of four systems in the form of (5) and with nonlinear dynamics $f(t, x_i) = 2\sin t \sin x_i$ and initial states $x(0) = [1\ 2\ 3\ 4]^T$. It follows from

$$\int \frac{dw}{\sin w} = \frac{1}{2} \ln \frac{1 - \cos w}{1 + \cos w}$$

that the uncontrolled systems are uniformly bounded.

Suppose that the network topology periodically switches according to the digraphs S_i in Fig. 1(a) and that all the communication channels are equally weighted. That is, digraphs S_i with $i = 1, 2, 3$ have a dwell time of 1 s and hold for time interval $[3k + i - 1,\ 3k + i)$ for $k = 0, 1, 2, \ldots$. Hence, the cumulative graph is strongly connected with $\overline{T} = 3$ s.

It is straightforward to show that no consensus can be achieved under input (7) but, as shown in Fig. 1(b), a consensus is reached at time $t = 3.2$ s under the proposed protocol of (14) and with $\alpha = 5$. □

Finite-time consensus control (14) depends upon the knowledge of γ_i and, in the subsequent section, a new distributed estimator is provided to generate this information.

5. Distributed connectivity estimation of directed network

The primary left eigenvector γ has several implications: should the topology be constant, it provides the consensus value of $\gamma^T x(0)$ under the standard protocol (7); its entries describe the physical interconnection of the nodes; and its knowledge can be used either to improve the asymptotic convergence (Qu et al., 2014) or to achieve the finite-time convergence as shown by Theorem 2. Estimation of γ can be done using the distributed observer in Lemma 1, and it requires resettings of local observer's initial conditions every time when a topological switching is detected locally or a resetting is propagated from one of the neighbors. Alternatively, we can use the distributed estimator proposed below, and this new estimation scheme does not need any resetting but requires synchronization of time in order to inject sinusoidal functions. To ensure fast convergence, the following assumption is introduced, and the rationale is that an online estimation and control scheme would not work if topology changes are arbitrarily fast.

Assumption 1b. Time sequence $\{t_k : k \in \aleph^+\}$ of topology changes has the property that $(t_{k+1} - t_k) \ge \underline{T}$ for some known constant $\underline{T} > 0$.

The proposed observer consists of two parts. The first part consists of an nth-order distributed observer: at the ith system, the observer state is defined by $\theta^{(i)} = [\theta_1^{(i)} \ldots \theta_n^{(i)}]^T \in \Re^n$, and evolution of the kth entry of $\theta^{(i)}$ is defined by

$$\dot{\theta}_k^{(i)} = \mu \sum_{j=1}^{n} a_{ij}[\theta_k^{(j)} - \theta_k^{(i)}] + \begin{cases} \sin \omega t & \text{if } k = i \\ \sin t & \text{if } k \ne i, \end{cases} \tag{16}$$

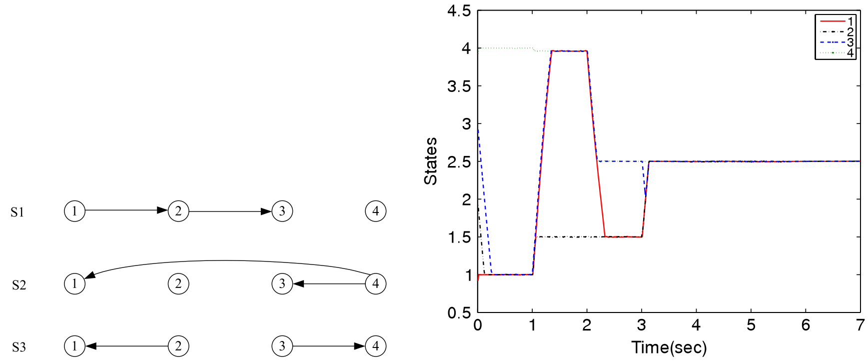
where $\mu > 1$ is the gain to be chosen, and integer $\omega \in \aleph^+$ is selected such that $\omega \underline{T} \ge 16\pi$. As the second part, the estimate of the ith entry of γ is locally calculated at the ith system and using the following formula:

$$\hat{\gamma}_i = \lim_{\mu \to \infty} \frac{\int_{t-\Delta}^{t} \left[\theta_i^{(i)}(\tau) + \cos \tau\right] \cos \omega\tau\, d\tau}{\int_{t-\Delta}^{t} \cos \tau \cos \omega\tau\, d\tau - \frac{\Delta}{2\omega}}, \tag{17}$$

where Δ is the period defined by

$$\Delta = \frac{2\kappa\pi}{\omega}, \quad \text{with } \kappa \in \aleph^+ \text{ and } \kappa \le \left\lfloor \frac{\omega \underline{T}}{16\pi} \right\rfloor, \tag{18}$$

and $\lfloor \cdot \rfloor$ denotes the floor operation.



(a) Graph. (b) Consensus.

Fig. 1. Example 1.

It follows from (16) that $\theta_k^{(i)}$ involves according to the same communication network described by A and that the corresponding observer is distributed. In addition, the proposed estimator contains two known, synchronized, and periodical perturbations whose frequencies are chosen according to (16) and (18). Performance of observer (16) and (17) is summarized into the following theorem.

Theorem 3. *Consider the graph Laplacian $\mathcal{L}$ defined in (2), and left eigenvector γ defined in (3). Then, under Assumption 1b, γ can be estimated locally using estimators defined in (16) and (17), provided that μ is chosen to be sufficiently large.*

Proof. It follows from (16) that

$$\dot{\theta}_i = -\mu\mathcal{L}\theta_i + (\mathbf{1}_n - e_i)\sin t + e_i \sin\omega t \tag{19}$$

where e_i is the unit vector defined in Lemma 2, and $\theta_i = [\theta_i^{(1)} \ldots \theta_i^{(n)}]^T$ is the overall estimator for the ith entry γ_i. The subsequent analysis is done for (19), $i = 1, \ldots, n$.

The solution to (19) is

$$\theta_i(t) = e^{-\mu\mathcal{L}(t-t_0)}\left\{\theta_i(t_0) + \int_{t_0}^{t} e^{\mu\mathcal{L}\tau}\left[(\mathbf{1}_n - e_i)\sin\tau + e_i\sin\omega\tau\right]d\tau\right\}. \tag{20}$$

Without loss of any generality, let us assume that $\mathcal{L}$ has at least one globally reachable node (otherwise, a permutation matrix can be used to transform $\mathcal{L}$ into a block diagonal matrix and the subsequent derivations can be applied to each of its diagonal sub-blocks). It follows from Lemma 2 that

$$\int e^{\mp\mu\mathcal{L}\tau}\sin\omega\tau\, d\tau + \frac{\cos\omega\tau}{\omega}\mathbf{1}_n\gamma^T = \Gamma_s(\mu^2\Lambda_s^2 + \omega^2 I)^{-1}(\mp\mu\Lambda_s\sin\omega\tau - I\omega\cos\omega\tau)\, e^{\mp\mu\Lambda_s\tau}W_s^T.$$

Substituting the above expression into (20) and utilizing the identities of $(\mathbf{1}_n\gamma^T)^2 = \mathbf{1}_n\gamma^T$, $(\Gamma_s e^{-\mu\Lambda_s t}W_s^T)(\mathbf{1}_n\gamma^T) = \mathbf{0}$, and $(\mathbf{1}_n\gamma^T)\Gamma_s = \mathbf{0}$, we have

$$\begin{aligned}\theta_i(t) = {} & \left(\mathbf{1}_n\gamma^T + \Gamma_s e^{-\mu\Lambda_s(t-t_0)}W_s^T\right)\theta_i(t_0) \\ & + \frac{\cos\omega t_0 - \cos\omega t}{\omega}\mathbf{1}_n\gamma^T e_i \\ & + (\cos t_0 - \cos t)\mathbf{1}_n\gamma^T(\mathbf{1}_n - e_i) + (\Gamma_s e^{-\mu\Lambda_s t}W_s^T) \\ & \times \Big\{\Gamma_s(\mu^2\Lambda_s^2 + I)^{-1}\big[(\mu\Lambda_s\sin t - I\cos t)\, e^{\mu\Lambda_s t} \\ & - (\mu\Lambda_s\sin t_0 - I\cos t_0)\, e^{\mu\Lambda_s t_0}\big]W_s^T(\mathbf{1}_n - e_i) \\ & + \Gamma_s(\mu^2\Lambda_s^2 + \omega^2 I)^{-1}\big[(\mu\Lambda_s\sin\omega t - I\cos\omega t)\, e^{\mu\Lambda_s t} \\ & - (\mu\Lambda_s\sin\omega t_0 - I\cos\omega t_0)\, e^{\mu\Lambda_s t_0}\big]W_s^T e_i\Big\}.\end{aligned}$$

Multiplying $\cos\omega t$ on both sides, integrating over the time interval $[t,\ t+\Delta]$ with Δ defined in (18), and recalling that

$$\int_{t-\Delta}^{t}\cos\omega\tau\, d\tau = 0 \quad\text{and}\quad \int_{t-\Delta}^{t}\cos^2\omega\tau\, d\tau = \frac{\Delta}{2},$$

we obtain that

$$\begin{aligned} & \int_{t-\Delta}^{t}\theta_i(\tau)\cos\omega\tau\, d\tau + \left(\int_{t-\Delta}^{t}\cos\tau\cos\omega\tau\, d\tau\right)\mathbf{1}_n\gamma^T(\mathbf{1}_n - e_i) \\ & = -\frac{\Delta}{2\omega}\mathbf{1}_n\gamma^T e_i + (\mu^2\Lambda_s^2 + \omega^2 I)^{-1}\big[\omega e^{-\mu\Lambda_s t}(e^{\mu\Lambda_s\Delta} - I)\sin\omega t \\ & \quad + \mu\Lambda_s e^{-\mu\Lambda_s t}(I - e^{\mu\Lambda_s\Delta})\cos\omega t\big] - \int_{t-\Delta}^{t}(\Gamma_s e^{-\mu\Lambda_s\tau}W_s^T) \\ & \quad \times \big[\Gamma_s(\mu^2\Lambda_s^2 + I)^{-1}(\mu\Lambda_s\sin\tau - I\cos\tau)\, e^{\mu\Lambda_s\tau}W_s^T(\mathbf{1}_n - e_i) \\ & \quad + \Gamma_s(\mu^2\Lambda_s^2 + \omega^2 I)^{-1}(\mu\Lambda_s\sin\omega\tau - I\cos\omega\tau) \\ & \quad \times e^{\mu\Lambda_s\tau}W_s^T e_i\big]\cos\omega\tau\, d\tau. \end{aligned} \tag{21}$$

It follows from Lemma 2 that, as $\mu \to \infty$,

$$\lim_{\mu\to\infty} e^{-\mu\Lambda_s t} = \mathbf{0} \quad\text{and}\quad \lim_{\mu\to\infty}(\mu^2\Lambda_s^2 + I)^{-1} = \mathbf{0}.$$

Consequently, for any t and for sufficiently large μ, Eq. (21) is reduced to

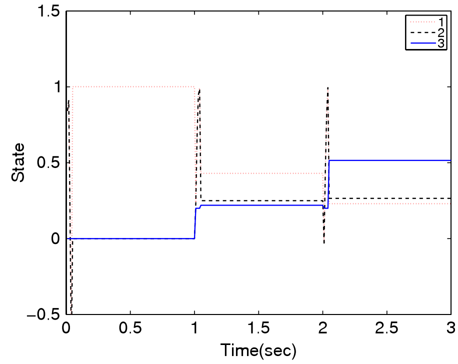
$$\begin{aligned} & \lim_{\mu\to\infty}\int_{t-\Delta}^{t}[\theta_i(\tau) + \cos\tau\,\mathbf{1}_n]\cos\omega\tau\, d\tau \\ & = \left(\int_{t-\Delta}^{t}\cos\tau\cos\omega\tau\, d\tau - \frac{\Delta}{2\omega}\right)\mathbf{1}_n\gamma^T e_i. \end{aligned}$$

In particular, the ith entry of θ_i can be calculated as

$$\begin{aligned} & \lim_{\mu\to\infty}\int_{t-\Delta}^{t}\left[\theta_i^{(i)}(\tau) + \cos\tau\right]\cos\omega\tau\, d\tau \\ & = \left(\int_{t-\Delta}^{t}\cos\tau\cos\omega\tau\, d\tau - \frac{\Delta}{2\omega}\right)\gamma_i, \end{aligned}$$

from which (17) is concluded to estimate the left eigenvector γ distributively. □

It follows that the convergence of (17) can be made faster by making μ sufficiently small, which allows smaller Δ, and indeed the proposed observer is naturally independent of any resetting of its initial conditions, hence no further action is required at either the local or network level when switching topologies are considered. Hence, output $\hat{\gamma}_i$ converges to the value of γ_i distributively and the convergence can be achieved no later than $T/4$ after each topology switching. Performance of the proposed observer is illustrated in the following example.



(a) System 1. (b) System 3.

Fig. 2. Estimation of first left eigenvector γ.

Example 2. Consider the time sequence $\{t_k : k \in \{3l+1,\ 3l+2,\ 3l+3\};\ l \in \aleph\}$ and suppose that $A(t_{3l+j}) = A_j$, where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0.4 & 0.6 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.5 & 0.5 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.1 & 0 & 0.9 \\ 0.4 & 0.6 & 0 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}.$$

Their corresponding first left eigenvectors are $\gamma(A_1) = [1\,0\,0]^T$, $\gamma(A_2) = [0.43\ 0.35\ 0.22]^T$, and $\gamma(A_3) = [0.23\ 0.26\ 0.51]^T$, respectively. The proposed distributed observer given by (16) and (17) is implemented with $\mu = 1000$, $\omega = 60$, and $\Delta = \pi/30$ sec. As shown in Fig. 2, the convergence performance of the observer is prompt and accurate.

6. Conclusion

Finite time consensus of a class of nonlinear systems is investigated under directed and switching topologies. A contraction mapping argument is used to study the convergence properties and to establish the connection between convergence and varying topologies. It is shown that, should the first left eigenvector is known or can be estimated locally, finite time consensus can be ensured if varying topologies either individually have a globally reachable node or over time are jointly strongly connected. In addition, a novel distributed observer is proposed to estimate the first left eigenvector so that the estimation-based control scheme not only is distributed but has finite-time convergence under switching topologies.

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